Indian Statistical Institute, Bangalore

M. Math. Second Year

First Semester - Number Theory

Mid-Semester Exam | Duration: 3 hours | Date : Sept 10, 2014

Note: Each question carries 20 marks. Ans as many questions as you can.Max Marks: 100

1. (a) Show that, for positive integers m and n, the g.c.d of $2^m - 1$ and $2^n - 1$ is $2^{(m,n)} - 1$, where (m, n) is the g.c.d of m and n.

   (b) If $2^n + 1$ is a prime for some n, then show that n is a power of 2.

2. (a) Let p_n denotes the n^{th} prime in natural order. Then show that $p_{n+1} \leq p_1 \cdots p_n + 1$.

   (b) Use part (a) to show that there is a constant $c > 0$ such that the prime counting function satisfies $\pi(x) \geq c \log \log x$ for all $x \geq 3$.

3. (a) Prove a necessary and sufficient condition on the odd prime p so that 2 is a square modulo p.

   (b) If $n \geq 2$, then show that any prime factor of the n^{th} Fermat number F_n is $\equiv 1(\text{mod} 2^{n+2})$.

4. (a) Let w be a complex cube root of unity. Then show that the ring $Z[w]$ is a U.F.D.

   (b) Find all the units of $Z[w]$.

   (c) Show that the ring $Z[\sqrt{5}]$ is not a U.F.D.

5. Prove that every rational number has exactly two simple continued fraction expansions.

6. Let $p_i, 1 \leq i \leq 5$, and $q_i, 1 \leq i \leq 5$, be the first five primes which are $1(\text{mod} 4)$ and $3(\text{mod} 4)$, respectively. Let A, B, C be the 5×5 matrices whose entries are the Legendre symbol $(\frac{p_i}{p_j}), (\frac{q_i}{q_j}), (\frac{p_i}{q_j})$, respectively. Compute these matrices.